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# Weakly normal subgroups and classes of finite groups

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**Abstract.** A subgroup  $K$  of a group  $G$  is said to be weakly normal in  $G$  if  $K^g \leq N_G(K)$  implies  $g \in N_G(K)$ . In this paper we establish certain characterizations of solvable  $PST$ -groups using some weakly normal subgroups.

**Keywords:** Weakly normal subgroups, solvable  $PST$ -groups

**MSC 2000 classification:** 20D10, 20D20

## 1 Introduction and statements of results.

All groups are finite.

A subgroup  $H$  of a group  $G$  is said to be weakly normal in  $G$  provided that if  $g \in G$  and  $H^g \leq N_G(H)$ , then  $g \in N_G(H)$ . It is not difficult to see that  $H$  is weakly normal in  $G$  if and only if whenever  $H \triangleleft \langle H, H^g \rangle$ , then  $H \triangleleft \langle H, g \rangle$ . For each group  $G$  let  $WN(G)$  denote the set of all weakly normal subgroups of  $G$ . Notice that if  $H$  is normal in  $G$  or  $N_G(H) = H$ , then  $H \in WN(G)$ . Thus  $WN(G)$  contains all maximal subgroups of  $G$ . Moreover,  $WN(G)$  contains all the pronormal subgroups of  $G$  but there are weakly normal subgroups of  $G$  which are not pronormal (see [2, p. 28]). Also the join of two pronormal subgroups of a group need not be pronormal but belongs to  $WN(G)$  (see [2, p. 28]).

In [5] the authors introduce the concept of a  $\mathcal{H}$ -subgroup of a group and prove a number of very interesting results about such subgroups. A subgroup  $X$  of a group  $G$  is called a  $\mathcal{H}$ -subgroup provided that  $X^g \cap N_G(X) \leq X$  for all  $g \in G$ . The authors of [5] denote by  $\mathcal{H}(G)$  the set of all  $\mathcal{H}$ -subgroups of  $G$ . Assume that  $X \in \mathcal{H}(G)$  and  $X^g \leq N_G(X)$ . Then  $g \in N_G(X)$  so that  $\mathcal{H}(G) \subseteq WN(G)$ . In particular, by Proposition 3 of [5], a Sylow subgroup of a normal subgroup of  $G$  belongs to  $WN(G)$ . Moreover, a Hall subgroup of  $G$  also is a  $WN$ -subgroup of  $G$ .

**Example 1.** ([3]). Let  $G = S_4$  and  $H = \langle (1\ 2\ 3\ 4) \rangle$ . Then  $N_G(H) = \langle (1\ 2\ 3\ 4), (1\ 3) \rangle$ , and if  $g = (1\ 2\ 3)$ ,  $H^g = \langle (1\ 4\ 2\ 3) \rangle$ , whence  $H^g \cap N_G(H) = \langle (1\ 2)(3\ 4) \rangle \not\leq H$ .

Thus  $H$  is not an  $\mathcal{H}$ -group of  $G$ . Note that  $N_G(H)$  has a unique cyclic group of order 4. Consequently, if  $H^x \leq N_G(H)$ ,  $x$  an element of  $G$ , then  $H^x = H$  and  $H$  is weakly normal in  $G$ .

However, Ballester-Bolinches and Esteban-Romero [3] proved the following very interesting results.

**Theorem 1.** *Let  $H$  be a weakly normal subgroup of the supersolvable group  $G$ . Then*

- (a) *If  $H$  is a  $p$ -group,  $p$  a prime, then  $H$  is an  $\mathcal{H}$ -subgroup of  $G$ .*
- (b) *If every subgroup of  $H$  is weakly normal in  $G$ , then  $H$  is an  $\mathcal{H}$ -subgroup of  $G$ .*

A group  $G$  is called a  $T$ -group if whenever  $H$  and  $K$  are subgroups of  $G$  such that  $H \triangleleft K \triangleleft G$ , then  $H \triangleleft G$ . W. Gaschütz [7] introduced the class of finite  $T$ -groups and characterized solvable  $T$ -groups. He proved that a solvable group  $G$  is a  $T$ -group if and only if the nilpotent residual  $L$  of  $G$  is a normal abelian Hall subgroup of  $G$  such that  $G$  acts by conjugation on  $L$  as power automorphisms and  $G/L$  is a Dedekind group.

Bianchi, Mauri, Herzog, and Verardi [5], Ballester-Bolinches and Esteban-Romero [3], and Csörgö and Herzog [6] provide a number of extensions of Gaschütz's [7] characterization of solvable  $T$ -groups.

**Theorem 2.** ([3, 5]). *Let  $G$  be a group. The following statements are equivalent:*

- (a)  *$G$  is a solvable  $T$ -group.*
- (b) *Every subgroup of  $G$  belongs to  $\mathcal{H}(G)$ .*
- (c) *Every subgroup of  $G$  belongs to  $WN(G)$ .*

**Theorem 3.** ([6]) *Let  $G$  be a solvable group. Then  $G$  is a  $T$ -group if and only if there exists a subgroup  $L$  of  $G$  such that*

- (a)  *$L$  is a normal Hall subgroup of  $G$ .*
- (b)  *$G/L$  is a Dedekind group.*
- (c) *Every subgroup of  $L$  of prime power order belongs to  $\mathcal{H}(G)$ .*

Throughout this paper  $S_p(G)$  denotes the set of cyclic subgroups of a group  $G$  of prime order or of order 4. Also  $\overline{S_p}(G)$  denotes the set of subgroups of  $G$  of prime power order. Note that  $S_p(G) \subseteq \overline{S_p}(G)$ .

**Theorem A.** Let  $G$  be a group. Then

- (a) If  $S_p(G) \subseteq WN(G)$ , then  $G$  is supersolvable.
- (b) If  $\overline{S_p}(G') \subseteq WN(G)$ , then  $G$  is supersolvable.

**Theorem B.** Let  $G$  be a group. Then  $S_p(G) \leq WN(G)$  if and only if there exist subgroups  $L$  and  $D$  of  $G$  such that

- (a)  $G = L \rtimes D$ , the semidirect product of  $L$  by  $D$ .
- (b)  $L$  and  $D$  are nilpotent Hall subgroups of  $G$ .
- (c)  $S_p(L) \subseteq WN(G)$ .
- (d)  $S_p(D) \subseteq WN(D)$ .

**Theorem C.** A group  $G$  is a solvable  $T$ -group if and only if  $G$  has subgroups  $L$  and  $D$  such that

- (a)  $G = L \rtimes D$ , the semidirect product of  $L$  by  $D$ .
- (b)  $L$  and  $D$  are nilpotent Hall subgroups of  $G$ .
- (c)  $\overline{S_p}(L) \subseteq WN(G)$ .
- (d)  $\overline{S_p}(D) \subseteq WN(D)$ .

One of the purposes of this paper is to determine if a theorem like Theorem C might be proven for some classes of groups related to solvable  $T$ -groups. Such classes include solvable  $PST$ - and  $PT$ -groups. Let  $G$  be a group. A subgroup  $H$  of  $G$  is said to permute with a subgroup  $K$  of  $G$  if  $HK$  is a subgroup of  $G$ .  $H$  is said to be  $S$ -permutable if it permutes with all the Sylow subgroups of  $G$ . Kegel [8] showed that an  $S$ -permutable subgroup of  $G$  is subnormal in  $G$ . Kegel's result generalized Ore's result [11] that a permutable subgroup of  $G$  is subnormal in  $G$ . A group  $G$  is called a  $PST$ -(resp.  $PT$ -)group if  $S$ -permutability (resp. permutability) is a transitive relation in  $G$ , that is, if  $H \subseteq K$  are subgroups of  $G$  such that  $H$  is  $S$ -permutative (resp. permutable) in  $K$  and  $K$  is  $S$ -permutable (resp. permutable) in  $G$ , then  $H$  is  $S$ -permutable (resp. permutable) in  $G$ . By Kegel's (resp. Ore's) result  $G$  is a  $PST$ -(resp.  $PT$ -)group if every subnormal subgroup of  $G$  is  $S$ -permutable (resp. permutable) in  $G$ .  $PST$ -groups have been studied in detail in [1, 2, 4].

Solvable  $PST$ - and  $PT$ -groups have been characterized by Agrawal [1]. He proved the following theorem.

**Theorem 4.** *Let  $G$  be a group. Then*

- (a)  $G$  is a solvable  $PST$ -group if and only if it has a normal abelian Hall subgroup  $L$  such that  $G/L$  is nilpotent and  $G$  acts by conjugation on  $L$  as power automorphisms.
- (b)  $G$  is a solvable  $PT$ -group if and only if it is a solvable  $PST$ -group with Iwasawa Sylow subgroups.

G. Zacher [13] proved part (b) of Theorem 4 in 1964. An Iwasawa group is one in which every subgroup is permutable.

**Theorem D.** A group  $G$  is a solvable  $PST$ -group if and only if it has subgroups  $L$  and  $D$  such that

- (a)  $G = L \rtimes D$ .
- (b)  $L$  and  $D$  are nilpotent Hall subgroups of  $G$ .
- (c)  $\overline{S_p}(L) \subseteq WN(G)$ .

**Theorem E.** Let  $L$  be a normal Hall subgroup of  $G$  such that

- (a)  $G/L$  is a  $PST$ -group.
- (b) Every subnormal subgroup of  $L$  is weakly normal in  $G$ .

Then  $G$  is a  $PST$ -group.

**Theorem F.** A group  $G$  is a solvable  $PST$ -group if and only if it has a normal nilpotent Hall subgroup  $L$  such that  $G/L$  is a solvable  $PST$ -group and  $\overline{S_p}(L) \subseteq WN(G)$ .

## 2 Preliminary results.

By Lemma 2.1 of [10] and Lemma 3 of [3] we have

**Lemma 1.** Let  $H$ ,  $K$  and  $N$  be subgroups of  $G$ . Then

- (a) If  $H \leq K$  and  $H \in WN(G)$ , then  $H \in WN(K)$ .
- (b) Let  $N \leq H$ . Then  $H \in WN(G)$  if and only if  $H/N \in WN(G/N)$ .
- (c) If  $H \trianglelefteq K$  and  $H \in WN(G)$ , then  $H \trianglelefteq K$ .

**Lemma 2.** Let  $G$  be a group.

- (a) Let  $M$  and  $L$  be subgroups of  $G$  such that  $ML = LM$ ,  $(|L|, |M|) = 1$  and  $G = MN_G(L)$ . Then  $L \in WN(G)$ .

- (b) Let  $N \triangleleft G$  and  $P$  a  $p$ -subgroup of  $G$  such that  $(p, |N|) = 1$ . If  $P \in WN(G)$ , then  $PN \in WN(G)$  and  $PN/N \in WN(G/N)$ .

*Proof.* The exact same proofs of Lemmas 5 and 6 of [6] establish parts (a) and (b) of Lemma 2.  $\square$  QED

The next lemma contains results established by Li in [9].

**Lemma 3.** Let  $G$  be a group,  $p$  a prime and  $P$  a Sylow  $p$ -subgroup of  $G$ .

- (a) If  $p > 2$  and every minimal subgroup of  $P$  lies in the center of  $N_G(P)$ , then  $G$  is  $p$ -nilpotent.
- (b) If  $p = 2$  and every cyclic subgroup of  $P$  of order 2 or 4 is normal in  $N_G(P)$ , then  $G$  is 2-nilpotent.
- (c) If  $G$  possesses a normal 2-complement  $N$  and if every minimal subgroup of any Sylow subgroup  $R$  of  $N$  is normal in  $N_G(R)$ , then  $G$  is supersolvable.

### 3 Proofs of the Theorems.

*Proof of Theorem A.* (a) Assume that  $S_p(G) \subseteq WN(G)$ . Let  $P$  be a Sylow  $p$ -subgroup of  $G$ ,  $p$  a prime and let  $U \leq P$  be a subgroup of order  $p$  or 4 if  $p = 2$ . Then  $U \in S_p(G) \subseteq WN(G)$  and  $U \triangleleft \triangleleft N_G(P)$ . By part (c) of Lemma 1  $U \triangleleft N_G(P)$ . Hence for every prime  $p$  the subgroups of  $S_p(P)$  are normal in  $N_G(P)$ . Thus  $G$  is supersolvable by parts (b) and (c) of Lemma 3. Thus (a) is true. We also note (a) follows from Theorem 3.1 of [10].

Assume that  $\overline{S_p(G')} \subseteq WN(G)$ . We may assume that  $G' \neq 1$ . Note that  $S_p(G') \subseteq \overline{S_p(G')} \subseteq WN(G')$  by part (a) of Lemma 1 and hence  $G'$  is supersolvable. Clearly  $G$  is solvable. Let  $M$  be a minimal normal subgroup of  $G$  contained in  $G'$ .  $M$  is an elementary abelian  $q$ -group for some prime  $q$ . Let  $\langle x \rangle$  be a subgroup of  $M$  of order  $q$ . Then  $\langle x \rangle \in WN(G)$  and by part (c) of Lemma 1  $\langle x \rangle \triangleleft G$ . Consider  $G/\langle x \rangle$  and note  $G'/\langle x \rangle = (G/\langle x \rangle)'$ . By part (b) of Lemma 1 and part (b) of Lemma 2  $\overline{S_p(G'/\langle x \rangle)} \leq WN(G/\langle x \rangle)$ . Hence, by induction,  $G/\langle x \rangle$  is supersolvable and so  $G$  is also supersolvable. Thus (b) is also true.  $\square$  QED

*Proof of Theorem B.* Assume that  $S_p(G) \subseteq WN(G)$ . By part (a) of Theorem A  $G$  is supersolvable. Thus  $S_p(G) \subseteq WN(G) \subseteq \mathcal{H}(G)$  by part (a) of Theorem 1. By Theorem 10 of [6] there are subgroups  $L$  and  $D$  of  $G$  which satisfy  $L$  and  $D$  are Hall nilpotent subgroups of  $G$  such that  $L \triangleleft G$  and  $G = L \rtimes D$ . Also  $S_p(L)$  consists of normal subgroups of  $G$  and  $S_p(D)$  consists of normal subgroups of  $D$ . Therefore, (a)–(d) holds.

Conversely, assume that  $G$  has Hall subgroups  $L$  and  $D$  which satisfy (a)–(d). By part (c) of Lemma 1 the subgroups in  $S_p(L)$  are all normal in  $G$  and the subgroups in  $S_p(D)$  are normal in  $D$ . Thus, by Theorem 10 of [6],  $S_p(G) \subseteq \mathcal{H}(G)$ . But  $\mathcal{H}(G) \subseteq WN(G)$ .

This completes the proof of Theorem B.  $\square QED$

*Proof of Theorem C.* Assume  $G$  satisfies (a)–(d) and let  $X$  be a  $q$ -subgroup of  $L$ ,  $q$  a prime. By (b)  $L$  is nilpotent and so  $X \triangleleft \triangleleft G$ . By (c)  $X \in WN(G)$  and hence, by part (c) of Lemma 1,  $X \triangleleft G$ . Thus  $L$  is a Dedekind group. Now let us assume  $X$  has order  $q$ . Then  $L/X$  is a Hall nilpotent subgroup of  $G/X$  and  $\overline{S_p}(L/X) \subseteq WN(G/X)$  by part (b) of Lemma 1. Also  $DX/X$  is a nilpotent Hall subgroup of  $G/X$  and it is isomorphic to  $D$ . Thus  $\overline{S_p}(DX/X) \subseteq WN(DX/X)$ . This means that  $G/X$  satisfies (a)–(d) with respect to  $L/X$  and  $DX/X$ . By induction on the order of  $G$ ,  $G/X$  is supersolvable and so  $G$  is supersolvable. This means  $L$  is abelian and  $G$  acts on  $L$  as power automorphisms. By Gaschütz's Theorem [7] it is enough to show  $D$  is a Dedekind group. By part (a) of Theorem 1  $\overline{S_p}(L) \subseteq \mathcal{H}(G)$  and  $\overline{S_p}(D) \subseteq \mathcal{H}(D)$ . Let  $r$  be a prime divisor of the order of  $D$  and let  $R$  be the Sylow  $r$ -subgroup of  $D$ . Note that  $\overline{S_p}(R) = \mathcal{H}(R)$ . Let  $Y$  be a subgroup of  $R$ . Then  $Y \triangleleft R$  by part (c) of Lemma 1. Thus  $D$  is a Dedekind group.

Conversely, assume that  $G$  is a solvable  $T$ -group. Then, by Gaschütz's Theorem [7], the nilpotent residual  $L$  of  $G$  is a normal abelian Hall subgroup of  $G$  on which  $G$  acts as power automorphisms and  $G/L$  is a Dedekind group. Let  $D$  be a system normalizer of  $G$ . By the Gaschütz, Shenkman and Carter Theorem, Theorem 9.2.7 of [12, p. 264],  $G = L \rtimes D$ . Thus (a) and (b) hold. Since all the subgroups of  $L$  are normal in  $G$  it follows that  $\overline{S_p}(L) \subseteq WN(G)$ . Likewise, since  $D$  is a Dedekind group,  $\overline{S_p}(D) \subseteq WN(D)$ . This completes the proof.  $\square QED$

*Proof of Theorem D.* Assume that  $G$  is a solvable  $PST$ -group and let  $L$  be the nilpotent residual of  $G$ . By part (a) of Theorem 4  $L$  is an abelian normal Hall subgroup of  $G$  on which  $G$  acts by conjugation as power automorphisms. Let  $D$  be a system normalizer of  $G$ . By Theorem 9.2.7 of [12, p. 264]  $G = L \rtimes D$ . Note that  $D$  is a nilpotent Hall subgroup of  $G$ . Let  $X$  be a subgroup of  $L$ . Then  $X \triangleleft G$  and so  $\overline{S_p}(L) \subseteq WN(G)$ . Therefore, (a), (b) and (c) are true.

Conversely, assume (a), (b) and (c) hold for the group  $G$ . Then  $G$  has nilpotent Hall subgroups  $L$  and  $D$  such that  $L \triangleleft G$  and  $G = L \rtimes D$ . Clearly  $G$  is solvable. Note that  $G/L$  is nilpotent and hence is a  $PST$ -group. Let  $X$  be a subgroup of  $L$  of prime power order. Then  $X \in \overline{S_p}(L) \subseteq WN(G)$  and  $X \triangleleft \triangleleft G$ . By part (c) of Lemma 1  $X \triangleleft G$ . Therefore, by Theorem 2.4 of [1]  $G$  is a solvable  $PST$ -group.  $\square QED$

*Proof of Theorem E.* Let  $L$  be a normal Hall subgroup of  $G$  such that  $G/L$  is a  $PST$ -group and every subnormal subgroup of  $L$  belongs to  $WN(G)$ . Let  $X$  be a subnormal subgroup of  $L$ . Then  $X$  is subnormal in  $G$  and hence  $X$  is normal in  $G$  by part (c) of Lemma 1. Thus  $G$  is a  $PST$ -group by Theorem 2.4 of [1].  $\square$

*Proof of Theorem F.* This follows from Theorem 2.3 of [1] and Theorem E.  $\square$

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